Homework 5 (Due 2/16/2014)

Math 622

February 18, 2014

1. Let $W(t)$ be a Brownian motion and $M(t) := \max_{s\in[0,t]} W(s)$ be its running maximum. Show that for $k < b$, $b > 0$

$$\Big\{k < W(s), M(s) > b\Big\} = \Big\{M(s) > b, B^{\tau_b}(s) < 2b - k\Big\}.$$

Complete the proof of the formula for $H(\alpha, 0, k, b)$ in Section 2.2 of *Notes to Lecture 5b: Reflection principle.*

2. Let $\widehat{W}$ be a Brownian motion, let $\widehat{M}$ be its running maximum and let $\widehat{H}(t) = \min_{s\in[0,t]}\{\widehat{W}(s)\}$ be its running minimum. Assume they are defined on a probability space $(\Omega, \mathcal{F}, \widehat{\mathbf{P}})$, and denote expectation on this space by $\widehat{E}[\cdot]$. Let $f_t(m, w)$ be the joint density of $(\widehat{M}(t), \widehat{W}(t))$. Because $-\widehat{W}(t)$ is a Brownian motion and $\widehat{H}(t) = -\max_{s\in[0,t]}\{-\widehat{W}(s)\}$ it follows that $(\widehat{H}(t), \widehat{W}(t))$ has the same distribution as $-(\widehat{M}(t), \widehat{W}(t))$.

The parts of this problem can be done independently of one another. As a guide to doing part b), read *Notes to Lecture 5b: Reflection principle.*

a) Show that the density of $(\widehat{H}(t), \widehat{W}(t))$ is

$$g_t(h, w) = f_t(-h, -w) = \begin{cases} 0, & \text{if } h > 0 \text{ or if } w < h; \\ \frac{2(w-2h)}{\sqrt{2\pi t^3}} e^{-(2h-w)^2/2t}, & \text{otherwise.} \end{cases}$$

b) Let $G(\alpha, k, b) := \widehat{E}\Big[\mathbf{1}_{\{\widehat{W}(\tau)\geq k\}}\mathbf{1}_{\{\widehat{H}(\tau)\geq b\}}e^{\alpha\widehat{W}(\tau)}\Big]$.

Let $0 < L < K$. Let $U(t)$ be the price at time $t < T$ of an option with payoff

$$(S(T) - K)^{+}\mathbf{1}_{\{\min\{S(u);u\leq T\}\geq L\}}.$$

This option is called a down-and-out call. Assume that $S(t)$ follows the Black-Scholes price model, $dS(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$, on a risk-neutral probability space $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}})$. Find a formula of the type

$$U(t) = \mathbf{1}_{\{\min\{S(u);0\le u\le t\}\ge L\}} u(t, S(t)),$$

and derive an explicit formula for $u(t,x)$ in terms of the function G.

c) We want to write G in terms of known functions. Let $H_s(\alpha, \beta, k, b)$ be the function defined in *Notes to Lecture 5b: Reflection principle.* Show that, for $0 > b$,

$$\begin{aligned} G(\alpha, k, b) &= \widehat{E}\left[e^{-\alpha \widehat{W}(\tau)}\right] - \widehat{E}\left[\mathbf{1}_{\{\widehat{W}(\tau) > -k\}} e^{-\alpha \widehat{W}(\tau)}\right] \\ &\qquad - H_\tau(-\alpha, 0, -\infty, -b) - H_\tau(-\alpha, 0, -k, -b) \\ &= e^{\alpha^2\tau/2} - e^{\alpha^2\tau/2} N\left(\frac{-\alpha\tau + k}{\sqrt{\tau}}\right) - H_\tau(-\alpha, 0, -\infty, -b) + H_\tau(-\alpha, 0, -k, -b). \end{aligned}$$

3. Let $S(t)$ follow the Black-Scholes model, $dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t)$. Let $Y(t) = \max\{S(u);\, u \le t\}$. Show that the joint process $(S(t), Y(t))$ is a Markov process.

4. Let $\{S(t);\, t \ge 0\}$ be a price process and let $Y(t) := \max\{S(u); 0 \le u \le t\}$. Consider an option whose payoff has the form $L(S(T), Y(T))$. We may think of any option with this type of payoff as a generalized lookback option. In specific cases, as we saw in class or in Shreve, there is an explicit formula for the price. However, any type of formula that simplifies the calculation of the price, by reducing it to operations that are easily implemented by numerical schemes, is useful. This exercise derives one such simplification for the Black-Scholes price model.

Assume that $dS(t) = rS(t)\,dt + \sigma\, d\widetilde{W}(t)$. Let

$$V(t) = e^{-r(T-t)} \widetilde{E}\Big[L(S(T), Y(T)) \big| \mathcal{F}(t)\Big]$$

be the price of the option with payoff $L(S(T), Y(T))$.

a) Find a function $G(x, y, w, m)$ such that $V(t) = v(t, S(t), Y(t))$, where

$$v(t, x, y) = E\Big[G\Big(x, y, \widehat{W}(T-t), \widehat{M}(T-t)\Big)\Big],$$

and where $\widehat{W}$ is a Brownian motion and $\widehat{M}$ is its running maximum.

b) Express $v(t, x, y)$ as a double integral.